

DIVIDED DIFFERENCES & RESTRICTION OPERATOR ON PALEY-WIENER SPACES PW_{τ}^p FOR N -CARLESON SEQUENCES

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ABSTRACT. For a sequence of complex numbers Λ we consider the restriction operator R_{Λ} defined on Paley-Wiener spaces PW_{τ}^p ($1 < p < \infty$). Lyubarskii and Seip gave necessary and sufficient conditions on Λ for R_{Λ} to be an isomorphism between PW_{τ}^p and a certain weighted l^p space. The Carleson condition appears to be necessary. We extend their result to N -Carleson sequences (finite unions of N disjoint Carleson sequences). More precisely, we give necessary and sufficient conditions for R_{Λ} to be an isomorphism between PW_{τ}^p and an appropriate sequence space involving divided differences.

1. INTRODUCTION

Let X be a Banach space of analytic functions defined on a domain Ω of the complex plane and Λ a sequence of points lying in Ω . The *restriction operator* R_{Λ} associated to Λ is defined on X by

$$R_{\Lambda} : X \ni f \mapsto (f(\lambda))_{\lambda \in \Lambda} \in \mathbb{C}^{\Lambda}.$$

Our aim is to describe the range of R_{Λ} , denoted by $X|\Lambda$, as well as the injectivity of R_{Λ} . This problem is related to interpolation problems in X and to geometrical properties of reproducing kernels in X^* . See [10], [16, Part D] or [18].

In the late 1950s and early 1960s, Carleson [4] ($p = \infty$) and Shapiro and Shields [19] ($1 \leq p < \infty$) showed that R_{Λ} is surjective from the Hardy space onto a suitable weighted l^p space if and only if Λ satisfies a certain separation condition, the so-called *Carleson* condition (more precise definitions below). Notice that, in Hardy spaces, as soon as the sequence satisfies the Blaschke condition, R_{Λ} cannot be injective.

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The results of Carleson and Shapiro-Shields have been generalized to finite unions of Carleson sequences (which are called N -Carleson sequences) by Vasyunin [21] ($p = \infty$) and Hartmann [8] ($1 < p < \infty$). A similar result has been obtained by Bruna, Nicolau and Øyma [3]. In this more general situation the description of the range of R_Λ involves divided differences.

Many authors like Hrushev, Nikolskii, Pavlov [10] or Minkin [13], have investigated interpolation problems in Paley-Wiener spaces using tools from operator theory (for instance invertibility criteria for a suitable Toeplitz operator) since the 1970s. Note that these spaces can be considered as special cases of backward shift invariant subspaces in Hardy spaces. More recently, Lyubarskii and Seip [12] have characterized the sequences Λ for which the associated restriction operator is an isomorphism between the Paley-Wiener space and an appropriate weighted l^p space. Their proof is in a sense more elementary and allows to consider sequences defined on the whole complex plane while the methods of Hrushev, Nikolskii, Pavlov intrinsically restrict the problem to sequences in a half-plane.

Here we investigate a generalization of Lyubarskii and Seip's result to N -Carleson sequences, in the spirit of Hartmann. Observe first that the Carleson condition turns out to be necessary for the classical interpolation problem in the Paley-Wiener space. Now, starting from an N -Carleson sequence Λ , we want to find necessary and sufficient conditions on Λ for R_Λ to be an isomorphism between the Paley-Wiener space and an appropriate sequence space involving now divided differences.

Let us fix the notation and the results we mentioned above. We first recall the definition of the *Hardy space*, for $1 \leq p < \infty$,

$$H^p(\mathbb{C}_a^\pm) := \left\{ f \in \text{Hol}(\mathbb{C}_a^\pm) : \sup_{y \geq a} \int_{\mathbb{R}} |f(x + iy)|^p dx < \infty \right\}$$

on the half-plane

$$\mathbb{C}_a^\pm := \{z \in \mathbb{C} : \text{Im}(z) \gtrless a\}, \quad (a \in \mathbb{R}).$$

For $p = \infty$,

$$H^\infty(\mathbb{C}_a^\pm) := \left\{ f \in \text{Hol}(\mathbb{C}_a^\pm) : \sup_{z \in \mathbb{C}_a^\pm} |f(z)| < \infty \right\}.$$

For short we will write $\mathbb{C}^\pm := \mathbb{C}_0^\pm$ and $H_\pm^p := H^p(\mathbb{C}^\pm)$. A function $I \in H^\infty(\mathbb{C}_a^\pm)$ satisfying $|I(x + ia)| = 1$ a.e. $x \in \mathbb{R}$ is called an *inner function*.

As previously mentioned, Carleson [4], Shapiro and Shields [19] solved the interpolation problem in the Hardy space. Their results were obtained in the unit disk, but translate clearly to any half-plane. Setting

$$l^p(|\operatorname{Im}(\lambda_n) - a|) := \left\{ u = (u_n)_{n \geq 1} : \sum_{n \geq 1} |\operatorname{Im}(\lambda_n) - a| |u_n|^p < \infty \right\},$$

we can state their result as follows. If $\Lambda = \{\lambda_n : n \geq 1\} \subset \mathbb{C}_a^\pm$, then

$$H^p(\mathbb{C}_a^\pm) |\Lambda = l^p(|\operatorname{Im}(\lambda_n) - a|)$$

if and only if Λ satisfies the *Carleson* condition

$$(1.1) \quad \inf_{\lambda \in \Lambda} \prod_{\substack{\mu \in \Lambda \\ \mu \neq \lambda}} \left| \frac{\lambda - \mu}{\lambda - \bar{\mu} - 2ia} \right| > 0.$$

Such sequences will be simply called *Carleson sequences*.

We consider now the *Paley-Wiener space* PW_τ^p (for $1 \leq p < \infty$) which consists of all entire functions of exponential type at most τ satisfying

$$\|f\|_p^p = \int_{\mathbb{R}} |f(x)|^p dx < \infty.$$

It is well-known (see e.g. [11]) that in the case $p = 2$, the Fourier transform is an isometric isomorphism between PW_τ^2 and $L^2(-\tau, \tau)$ which allows to reformulate the problem in terms of geometrical properties of exponentials in L^2 (we still refer to [10]). From the Plancherel-Polyà inequality (see Proposition 20 below), it follows that $PW_\tau^p = e^{-i\tau \cdot} K_{I^\tau}^p$, where

$$K_{I^\tau}^p := H_+^p \cap \overline{I^\tau} H_-^p$$

is the backward shift invariant subspace associated with the inner function $I^\tau(z) := \exp(2i\tau z)$, $z \in \mathbb{C}^+$. In particular, the Paley-Wiener space can be considered as a subspace of the Hardy space.

Luybarskii and Seip [12] gave necessary and sufficient conditions for R_Λ to be an isomorphism from PW_τ^p onto the weighted sequence space $l^p(e^{-p\tau|\operatorname{Im}(\lambda_n)|}(1 + |\operatorname{Im}(\lambda_n)|))$. Their proof is based on the boundedness of the Hilbert transform in certain weighted Hardy space.

Recall that the *Hilbert transform* \mathcal{H} is defined by

$$(1.2) \quad \mathcal{H}f(z) = \int_{-\infty}^{+\infty} \frac{f(t)}{t - z} dt,$$

where the integral has to be understood as a principle value integral for real z . It is known (see e.g [9] and [5]) that, if $w > 0$, \mathcal{H} is bounded

from the weighted space

$$L^p(w) := \left\{ f \text{ meas. on } \mathbb{R} : \int_{\mathbb{R}} |f|^p w dm < \infty \right\}$$

into itself, if and only if w satisfies the *Muckenhoupt* (A_p) condition

$$(A_p) \quad \sup_I \left(\frac{1}{|I|} \int_I w \right) \left(\frac{1}{|I|} \int_I w^{-\frac{1}{p-1}} \right)^{\frac{p}{p-1}} < \infty,$$

where the supremum is taken over all intervals of finite length. In [12], the authors also introduce the *discrete Hilbert transform* as follows. For fixed $\epsilon > 0$ and two sequences $\Gamma := \{\gamma_n\}_n$ and $\Sigma := \{\sigma_n\}_n$ satisfying $|\gamma_n - \sigma_n| = \epsilon$, and $a = (a_n)_n$,

$$(\mathcal{H}_{\Gamma, \Sigma}(a))_n := \sum_j \frac{a_j}{\gamma_j - \sigma_n}.$$

According to [12, Lemma 1]), $\mathcal{H}_{\Gamma, \Sigma}$ is bounded from $l^p(w_n)$ into itself if and only if $(w_n)_n$ satisfies the *discrete Muckenhoupt condition*

$$(\mathfrak{A}_p) \quad \sup_{\substack{k \in \mathbb{Z} \\ n > 0}} \left(\frac{1}{n} \sum_{j=k+1}^{k+n} w_j \right) \left(\frac{1}{n} \sum_{j=k+1}^{k+n} w_j^{-1/(p-1)} \right)^{p-1} < \infty.$$

Definition 1. A sequence $\Lambda \subset \mathbb{C}$ satisfies the condition $(LS)_{\tau, p}$ for $\tau > 0$ and $1 < p < \infty$, if the following set of conditions hold:

- (i) $\forall a \in \mathbb{R}$, $\Lambda \cap \mathbb{C}_a^\pm$ satisfies the Carleson condition (1.1);
- (ii) The sequence is *relatively dense*: $\exists r > 0$, $\forall x \in \mathbb{R}$,

$$d(x, \Lambda) := \inf_{\lambda \in \Lambda} |x - \lambda| < r;$$

- (iii) The limit

$$S(z) = \lim_{R \rightarrow \infty} \prod_{|\lambda| < R} \left(1 - \frac{z}{\lambda} \right)$$

exists and defines an entire function of exponential type τ ;

- (iv) The function $x \mapsto \left(\frac{|S(x)|}{d(x, \Lambda)} \right)^p$ satisfies (A_p) .

Note that if $0 \in \Lambda$, then the corresponding factor in (iii) reduces to z . In order to not complicate the notation we shall assume in all what follows that $0 \notin \Lambda$ which we can do without loss of generality (for instance, by shifting the sequence). We are now in a position to state the Lyubarski-Seip theorem [12, Theorem 1].

Theorem 2. (*Lyubarskii-Seip*). *Let $\Lambda \subset \mathbb{C}$, $\tau > 0$ and $1 < p < \infty$. The following assertions are equivalent.*

- (1) R_Λ is an isomorphism from PW_τ^p onto $l^p(e^{-p\tau|\text{Im}(\lambda)|} (1 + |\text{Im}(\lambda)|))$;
- (2) Λ satisfies $(LS)_{\tau, p}$.

Remark 3. The condition (iv) can be replaced by the condition (iv)'

(iv)' There is a relatively dense subsequence $\Gamma = (\gamma_n)_n \subset \Lambda$ such that the sequence $(|S'(\gamma_n)|^p)_n$ satisfies the discrete Muckenhoupt condition (\mathfrak{A}_p) .

The aim of this paper is to generalize the Lyubarskii-Seip result to finite unions of Carleson sequences. In the case of Hardy spaces, this problem has been solved by Vasyunin [21] and Hartmann [8] and involves divided differences.

As mentioned previously, in the case $p = 2$ the Fourier transform allows to express our main result Theorem 17 in terms of bases of exponentials in L^2 thereby generalizing a result by Avdonin and Ivanov [2, Theorem 3].

This paper is organized as follows. The next section will be devoted to divided differences. Section 3 deals with N -Carleson sequences. We will state our main result after some technical constructions in the fourth section. For an easier reading, we have postponed the proofs of Section 4 to the fifth section. Finally, in the last section we will discuss the necessity of the N -Carleson condition with an appropriate definition of the trace $PW_\tau^p|\Lambda$.

A final word on notation. If δ is a metric on Ω , we will denote by $D_\delta(x, \eta)$ the ball (relatively to δ) with center $x \in \Omega$ and radius $\eta > 0$, and $\text{diam}_\delta(E)$ the δ -diameter of E . We shortly write $\text{diam}(E)$ and $D(x, \eta)$ when δ is the Euclidian distance. If $\omega = (\omega_n)_{n \geq 1}$ is a sequence of strictly positive numbers and $1 \leq p < \infty$, we denote by $l^p(\omega)$ or $l^p(\omega_n)$ the space

$$l^p(\omega) := \left\{ a = (a_n)_{n \geq 1} : \sum_{n \geq 1} |a_n|^p \omega_n < \infty \right\}.$$

2. DIVIDED DIFFERENCES

Divided differences appear in many results about interpolation or bases of exponentials (see e.g. [21], [8], [3] or [2]). Here we will give the definitions and some properties that we will need later on. We recall that the (non-normalized) Blaschke factors in a half-plane \mathbb{C}_a^\pm are given by

$$b_\mu^{\pm, a}(z) = \frac{z - \mu}{z - \bar{\mu} - 2ia}.$$

(The formula is actually the same for the upper and the lower half-plane). The associated *pseudohyperbolic distance* will be denoted by

$$\rho_{\pm,a}(z, \mu) := |b_{\mu}^{\pm,a}(z)|.$$

For \mathbb{C}^+ , we will write $b_{\mu} = b_{\mu}^{+,0}$ and use ρ for $\rho_{+,0}$ and $\rho_{-,0}$.

The definitions and properties below are stated and proved in \mathbb{C}^+ but are obviously valid for any half-plane \mathbb{C}_a^{\pm} .

Definition 4. Let $\Gamma := \{\mu_i : 1 \leq i \leq |\Gamma| < \infty\} \subset \mathbb{C}^+$. For a finite set $a = \{a_i\}_{1 \leq i \leq |\Gamma|}$, we define the sequence of (*pseudohyperbolic divided differences*) of a relatively to Γ as follows

$$\Delta_{\Gamma}^0(a_i) := a_i, \quad \Delta_{\Gamma}^1(a_i, a_j) := \frac{a_j - a_i}{b_{\mu_i}(\mu_j)},$$

and

$$\Delta_{\Gamma}^k(a_{i_1}, \dots, a_{i_{k+1}}) := \frac{\Delta_{\Gamma}^{k-1}(a_{i_1}, \dots, a_{i_{k-1}}, a_{i_{k+1}}) - \Delta_{\Gamma}^{k-1}(a_{i_1}, \dots, a_{i_k})}{b_{\mu_{i_k}}(\mu_{i_{k+1}})}.$$

We will need to estimate the divided differences when Γ lies in a compact set $K \subset \mathbb{C}^+$ and $a = \{f(\mu) : \mu \in \Gamma\}$ for f an analytic function bounded in K . Here K is supposed to be the closure of a non empty open connected set. By $f \in H^{\infty}(K)$ we mean that f is holomorphic in the interior of K and

$$\|f\|_{\infty, K} := \sup_{z \in K} |f(z)| < \infty.$$

Lemma 5. Suppose that Γ lies in a compact set K with the properties mentioned above, and assume that there exists $\eta > 0$ such that $\rho(\Gamma, \partial K) \geq \eta$. Then, for each function $f \in H^{\infty}(K)$, we have

$$|\Delta_{\Gamma}^j(f(\mu^{(j+1)}))| \leq \left(\frac{2}{\eta}\right)^j \prod_{k=0}^j \left(\frac{1}{1 - \frac{k}{2M}}\right) \|f\|_{\infty, K}$$

where

$$\mu^{(j+1)} = (\mu_1, \dots, \mu_{j+1}) \text{ and } f(\mu^{(j+1)}) = (f(\mu_1), \dots, f(\mu_{j+1})).$$

Proof. Set

$$A_j := \left\{ z \in K : \rho(z, \partial K) \geq \frac{j}{2N} \eta \right\}, \quad 0 \leq j \leq N-1.$$

We show by induction over j that for every $z \in A_j$,

$$|\Delta_{\Gamma}^j(f(\mu^{(j)}, z))| \leq c_j \|f\|_{\infty, K}$$

with

$$c_j = \left(\frac{2}{\eta} \right)^j \prod_{k=0}^j \left(\frac{1}{1 - \frac{k}{2M}} \right).$$

Since $\Gamma \subset A_{N-1} \subset \dots \subset A_1 \subset A_0$, the result will follow. The claim is obviously true for $j = 0$. Now, the function

$$z \mapsto \Delta_\Gamma^{j+1} (f(\mu^{(j+1)}, z))$$

is holomorphic on A_{j+1} and by the maximum principle and the definition of divided differences, we have for $z \in A_{j+1}$,

$$(2.1) \quad |\Delta_\Gamma^{j+1} (f(\mu^{(j+1)}, z))| \leq \sup_{\xi \in \partial A_{j+1}} \left| \frac{\Delta_\Gamma^j (f(\mu^{(j)}, \xi)) - \Delta_\Gamma^j (f(\mu^{(j+1)}))}{\rho(\xi, \mu_{j+1})} \right|.$$

Let $\xi \in \partial A_{j+1}$. It is possible to find a point $\zeta \in \partial K$ such that

$$\rho(\zeta, \xi) = \left(\frac{j+1}{2N} \right) \eta$$

and so, since $\mu_{j+1} \in \Gamma$ and $\rho(\Gamma, \partial K) \geq \eta$, we have, by the triangle inequality,

$$(2.2) \quad \rho(\xi, \mu_{j+1}) \geq \rho(\zeta, \mu_{j+1}) - \rho(\xi, \zeta) \geq \eta \left(1 - \frac{j+1}{2N} \right).$$

From (2.1), (2.2) and the induction hypothesis, we finally obtain

$$|\Delta_\Gamma^{j+1} (f(\mu^{(j+1)}, \xi))| \leq \frac{2}{\eta} \left(\frac{1}{1 - \frac{j+1}{2N}} \right) c_j \|f\|_{\infty, K}$$

which gives the required estimate. \square

The next lemma will be important in the sequel; we can define a rational Newton type interpolating function which interpolates the values $\{a(\mu) : \mu \in \Gamma\}$ on Γ .

Lemma 6. *The holomorphic function*

$$P_{\Gamma, a}(z) := \sum_{k=1}^{|\Gamma|} \Delta_\Gamma^{k-1} (a(\mu^{(k)})) \prod_{l=1}^{k-1} b_{\mu_l}(z)$$

satisfies

$$P_{\Gamma, a}(\mu) = a(\mu), \quad \mu \in \Gamma.$$

The proof is quite straightforward (see also [7, p.80]).

Remark 7. Divided differences with respect to pseudohyperbolic metric can be found in [3, 8, 21]. We will also need *euclidian divided differences*:

$$\square_{\Gamma}^0 := a_i, \quad \square_{\Gamma}^1(a_i, a_j) := \frac{a_j - a_i}{\mu_j - \mu_i},$$

and

$$\square_{\Gamma}^k(a_{i_1}, \dots, a_{i_{k+1}}) := \frac{\square_{\Gamma}^{k-1}(a_{i_1}, \dots, a_{i_{k-1}}, a_{i_{k+1}}) - \square_{\Gamma}^{k-1}(a_{i_1}, \dots, a_{i_k})}{\mu_{k+1} - \mu_k}.$$

3. N -CARLESON SEQUENCES

Definition 8. Let $N \geq 1$ be a natural number. A sequence $\Lambda \subset \mathbb{C}_a^{\pm}$ is called a N -Carleson sequence if it is possible to find a partition

$$\Lambda = \bigcup_{i=1}^N \Lambda^i$$

such that, for every $i = 1, \dots, N$, the sequence Λ^i satisfies the Carleson (1.1) condition in \mathbb{C}_a^{\pm} .

Note that the number N is not uniquely defined.

Let us make a link between the N -Carleson condition and the *Generalized Carleson condition*, also called Carleson-Vasyunin condition (see e.g. [14] and references therein). The following result has originally been stated in \mathbb{D} (see [8, Proposition 3.1]) but can easily be translated to any half-plane \mathbb{C}_a^{\pm} .

Proposition 9. Let Λ be a sequence of complex numbers, lying in \mathbb{C}_a^{\pm} . The following assertions are equivalent

- (i) Λ is N -Carleson in \mathbb{C}_a^{\pm} ;
- (ii) There exists $\delta > 0$ and a sequence of Blaschke products $(B_n)_{n \geq 1}$ such that $\sup_n \deg B_n \leq N$, $\Lambda = \bigcup_n \sigma_n$, with $\sigma_n := \{\lambda \in \mathbb{C}_a^{\pm} : B_n(\lambda) = 0\}$ and $(B_n)_{n \geq 1}$ satisfies the Generalized Carleson condition

$$(3.1) \quad |B(z)| > \delta \inf_{n \geq 1} |B_n(z)|, \quad z \in \mathbb{C}_a^{\pm},$$

where B denotes the Blaschke product associated to Λ .

Observe that if Λ satisfies (ii), then, for $(\lambda, \mu) \in \sigma_n \times \sigma_m$ ($n \neq m$), we have $\rho(\sigma_n, \sigma_m) \geq \delta$ and thus

$$\inf_{n \neq m} \rho(\sigma_n, \sigma_m) \geq \delta > 0.$$

Remark 10. The subsets σ_n can for instance be obtained as intersections $\tau_n^\epsilon \cap \Lambda$ where τ_n^ϵ are the connected components of $L(B, \epsilon) := \{z : |B(z)| < \epsilon\}$ and ϵ is small enough. Moreover, choosing ϵ in a suitable way, it is possible to assume that the pseudohyperbolic diameter of σ_n is arbitrarily small.

Proposition 11. *Let $\Lambda = \{\lambda_n : n \geq 1\}$ be an N -Carleson sequence in \mathbb{C}_a^\pm . There exists $\eta > 0$ such that every connected component of $\bigcup_{n \geq 1} D_\rho(\lambda_n, \eta)$ admits at most N elements.*

Remark 12. We can deduce from the previous proposition that if Λ is N -Carleson in \mathbb{C}_a^\pm (or equivalently satisfies condition (ii) of Proposition 9), it is possible to construct a sequence of rectangles of \mathbb{C}_a^\pm defined by

$$R_n = \text{Rect}(z_n, L_n, l_n) = \left\{ x + iy \in \mathbb{C}_a^\pm : |x - x_n| \leq \frac{L_n}{2}, |y - y_n| \leq \frac{l_n}{2} \right\}$$

with $L_n, l_n > 0$ and $z_n = x_n + iy_n$. These rectangles satisfy the following properties:

$$(3.2) \quad \sigma_n \subset R_n, \quad n \geq 1;$$

$$(3.3) \quad L_n \asymp l_n \asymp |y_n - a| \asymp d(\partial R_n, \mathbb{R} + ia), \quad n \geq 1;$$

$$(3.4) \quad 0 < \inf_{n \geq 1} \rho(\sigma_n, \partial R_n) \leq \sup_{\substack{n \geq 1 \\ \lambda \in \sigma_n}} \rho(\lambda, \partial R_n) < \infty;$$

and finally, since the diameter of σ_n can be chosen arbitrarily small by Remark 10, we can suppose the R_n disjoint and even

$$(3.5) \quad \inf_{n \neq k} \rho(R_n, R_k) > 0.$$

Let Λ be N -Carleson in \mathbb{C}_a^\pm and $1 < p < \infty$. From Proposition 9, we can write

$$\Lambda = \bigcup_{n \geq 1} \sigma_n,$$

with in particular $|\sigma_n| \leq N$. We will construct divided differences relatively to σ_n . We set

$$\sigma_n = \{\lambda_{n,k} : 1 \leq k \leq |\sigma_n|\} \text{ and } \lambda_n^{(k)} = (\lambda_{n,1}, \dots, \lambda_{n,k}).$$

We choose, in an arbitrarily way, $\lambda_{n,0}$ in σ_n and introduce, for $a = (a(\lambda))_{\lambda \in \Lambda} \in \mathbb{C}^\Lambda$,

$$\|a\|_{X_{\pm a}^p(\Lambda)} := \left(\sum_{n \geq 1} |\operatorname{Im}(\lambda_{n,0}) - a| \sum_{k=1}^{|\sigma_n|} |\Delta_{\sigma_n}^{k-1}(a(\lambda_n^{(k)}))|^p \right)^{\frac{1}{p}}$$

and the space

$$X_{\pm a}^p(\Lambda) := \left\{ a \in \mathbb{C}^\Lambda : \|a\|_{X_{\pm a}^p(\Lambda)} < \infty \right\}.$$

Observe that for every $\lambda \in \sigma_n$, $1 \asymp |\operatorname{Im}(\lambda) - a| / |\operatorname{Im}(\lambda_{n,0}) - a|$ and so the definition of $X_{\pm a}^p(\Lambda)$ does not depend on the choice of $\lambda_{n,0}$. The following result was originally stated in \mathbb{D} (see [8]) but it is not hard to check that it holds in \mathbb{C}_a^\pm . The reader will find details in [6, p. 92].

Theorem 13. (Hartmann). *Let Λ be N -Carleson in \mathbb{C}_a^\pm and $1 < p < \infty$. Then, R_Λ is continuous and surjective from $H^p(\mathbb{C}_a^\pm)$ onto $X_{\pm a}^p(\Lambda)$.*

4. MAIN RESULT

Let Λ be a sequence in the complex plane. In this section we assume that there is an integer $N \geq 1$ such that for every $a \in \mathbb{R}$, the sequence

$$\Lambda_a^\pm := \Lambda \cap \mathbb{C}_a^\pm$$

is N -Carleson in the corresponding half-plane. Note that the partitions discussed in the previous section were adapted to sequences in a half-plane. Here, we will start discussing a “right” partition of Λ taking into account the fact that Λ lies in the whole complex plane

4.1. An adapted partition. From our above discussions it is possible to write

$$\Lambda_a^\pm = \bigcup_{n \geq 1} \sigma_{n,a}^\pm,$$

where $(B_{\sigma_{n,a}^\pm}^{\pm,a})_n$ satisfies the generalized Carleson condition in the corresponding half-plane \mathbb{C}_a^\pm ($B_{\sigma_{n,a}^\pm}^{\pm,a}$ being the Blaschke product in \mathbb{C}_a^\pm vanishing on $\sigma_{n,a}^\pm$). To simplify the notation, we will omit a if $a = 0$ and write

$$\sigma_n := \begin{cases} \sigma_{n+1}^+, & n \geq 0 \\ \sigma_n^-, & n < 0 \end{cases}.$$

The reader might notice that σ_n^+ and σ_m^- can come very close for certain values of n and m . This issue will be fixed below. Let us distinguish

the sets of points close to the real axis and the ones far away from it. Let us fix $\epsilon > 0$ for all what follows. We can assume that

$$\rho_0 := \sup_{n \in \mathbb{Z}} \text{diam}_\rho (\sigma_n) < \frac{\epsilon}{2}.$$

(Observe that ρ_0 is well defined by the Generalized Carleson condition). Next introduce

$$M_{\epsilon, \infty} := \{n \in \mathbb{Z} : \sigma_n \cap \{|\text{Im}(z)| < \epsilon\} = \emptyset\},$$

$$\Lambda_{\epsilon, \infty} := \bigcup_{n \in M_{\epsilon, \infty}} \sigma_n$$

(corresponding to the points for which the corresponding set σ_n does not intersect the previous strip) and

$$\Lambda_\epsilon := \Lambda \setminus \Lambda_{\epsilon, \infty}.$$

Notice that Λ_ϵ contains the points of Λ lying in the real axis and moreover

$$\Lambda_\epsilon \subset \{z \in \mathbb{C} : |\text{Im}(z)| < 3\epsilon\}.$$

Indeed, if $\lambda \in \Lambda_\epsilon$ and $\lambda \notin \mathbb{R}$, then there is $n_\lambda \in \mathbb{Z} \setminus M_{\epsilon, \infty}$ such that $\lambda \in \sigma_{n_\lambda}$. Hence, it is possible to find $\mu \in \sigma_{n_\lambda}$ such that $|\text{Im}(\mu)| < \epsilon$. It follows that

$$\begin{aligned} |\lambda - \mu| &= \frac{|\lambda - \mu|}{|\lambda - \bar{\mu}|} |\lambda - \bar{\mu}| \\ &\leq \rho_0 (2 |\text{Im}(\mu)|) + |\lambda - \mu| \\ &\leq \frac{3}{2} \epsilon^2 < \frac{3}{2} \epsilon, \end{aligned}$$

which implies that $|\text{Im}(\lambda)| < 5\epsilon/2$. Now, since Λ_ϵ is contained in a strip, parallel to the real axis, of finite width and is N -Carleson in $\mathbb{C}_{-3\epsilon}^+$, Λ_ϵ breaks up into a disjoint union

$$\Lambda_\epsilon = \bigcup_{n \geq 1} \sigma'_n$$

with

$$\rho'_0 := \sup_{n \geq 1} \text{diam}(\sigma'_n) < \frac{\epsilon}{2}$$

and moreover, for some $\delta > 0$, the subsets

$$\Omega_n := \left\{ z \in \mathbb{C} : \prod_{\lambda \in \sigma'_n} |z - \lambda| \leq \delta \right\}, \quad n \geq 1,$$

satisfy

$$(4.1) \quad \inf_{n \neq m} d(\Omega_n, \Omega_m) > 0.$$

This is possible in view of Remarks 10 and 12. It follows that we can write Λ as the following disjoint union

$$\Lambda = \left(\bigcup_{n \in M_{\epsilon, \infty}} \sigma_n \right) \cup \left(\bigcup_{n \geq 1} \sigma'_n \right) =: \bigcup_{n \in \mathbb{Z}} \tau_n.$$

Now that the partition is done, it is possible to construct divided differences. Since we will need both definitions of divided differences, we set

$$\tilde{\Delta}_{\tau_n} := \begin{cases} \Delta_{\tau_n} & \text{if } \exists k \text{ s.t. } \tau_n = \sigma_k \\ \square_{\tau_n} & \text{if } \exists k \text{ s.t. } \tau_n = \sigma'_k \end{cases}.$$

It is now possible to introduce a space of sequences that will be, assuming some hypotheses on Λ , the range of R_Λ . Naturally, we write

$$\tau_n = \{\lambda_{n,k} : 1 \leq k \leq |\sigma_n|\} \text{ and } \lambda_n^{(k)} := (\lambda_{n,1}, \dots, \lambda_{n,k}).$$

As previously, we choose, in an arbitrarily way, $\lambda_{n,0} \in \tau_n$, for every $n \in \mathbb{Z}$. We define, for $1 < p < \infty$,

$$X_{\tau, \epsilon}^p(\Lambda) := \left\{ a = (a(\lambda))_{\lambda \in \Lambda} : \|a\|_{X_{\tau, \epsilon}^p(\Lambda)} < \infty \right\},$$

with

$$\|a\|_{X_{\tau, \epsilon}^p(\Lambda)}^p := \sum_{n \in \mathbb{Z}} (1 + |\operatorname{Im}(\lambda_{n,0})|) \sum_{k=1}^{|\tau_n|} \left| \tilde{\Delta}_{\tau_n}^{k-1} (ae^{\pm i\tau \cdot} (\lambda_n^{(k)})) \right|^p,$$

and

$$e^{\pm i\tau \lambda} = \begin{cases} e^{i\tau \lambda} & \text{if } \lambda \in \tau_n, \quad n \in N_+, \\ e^{-i\tau \lambda} & \text{if } \lambda \in \tau_n, \quad n \in N_-, \end{cases}$$

where

$$N_+ := \{n \in \mathbb{Z} : \tau_n \cap (\mathbb{C}^+ \cup \mathbb{R}) \neq \emptyset\}$$

and

$$N_- := \mathbb{Z} \setminus N_+.$$

(The factor $e^{\pm i\tau \lambda}$ does not really matter close to \mathbb{R} .) Next proposition will be proved in Section 5.

Proposition 14. *If there exists $\epsilon > 0$ such that R_Λ is an isomorphism between PW_τ^p and $X_{\tau, \epsilon}^p(\Lambda)$ then Λ is relatively dense, i.e. there exists $r > 0$ such that for every $x \in \mathbb{R}$, $d(x, \Lambda) < r$.*

It follows from the conclusion of the previous proposition that the relative density is necessary. Thus, we will assume in all what follows that Λ is relatively dense:

$$\exists r > 0, \forall x \in \mathbb{R}, d(x, \Lambda) < r.$$

Still relative to the previous partition of Λ , we introduce, for $n \geq 1$, the products

$$p_n(x) := \prod_{\lambda \in \tau_n} |x - \lambda|$$

which permit us to define the function

$$d_N(x) := \inf_{n \in \mathbb{Z}} p_n(x), \quad x \in \mathbb{R}.$$

Remark 15. From the definition of the function d_N , we can do the following observations.

- (1) The relative density condition implies that

$$\sup_{x \in \mathbb{R}} d_N(x) \leq \left(r + \delta'_0 \right)^N < \infty,$$

where

$$\delta'_0 := \inf_{n \neq m} d(\sigma'_n, \sigma'_m) > 0.$$

- (2) It is clear that, in the definition of d_N , the infimum is actually a minimum. So, for each $x \in \mathbb{R}$, there is $n_x \in \mathbb{Z}$ such that $d_N(x) = p_{n_x}(x)$. It is not difficult to see that

$$\inf_{x \in \mathbb{R}} \inf_{m \neq n_x} p_m(x) \geq \left(\frac{\delta'_0}{2} \right)^N > 0.$$

- (3) Using the relative density, a similar reasoning as the one that can be used to show (2) yields that, with an other partition (and in particular with an other choice of ϵ), the function obtained is equivalent to d_N .

4.2. The theorem.

Definition 16. Let Λ be N -Carleson in every half-plane and relatively dense. We say that Λ satisfies the conditions $(H_N)_{\tau,p}$ (for $\tau > 0$ and $1 < p < \infty$) if

- (i) The limit

$$S(z) := \lim_{R \rightarrow \infty} \prod_{|\lambda| < R} \left(1 - \frac{z}{\lambda} \right)$$

exists and defines an entire function of exponential type τ .

- (ii) The function $x \mapsto \left(\frac{|S(x)|}{d_N(x)}\right)^p$ satisfies the (continuous) Muckenhoupt condition (A_p) .

The reader would notice that, in view of Remark 15–(3), the definition of the conditions $(H_N)_{\tau,p}$ do no depend on the partition of Λ .

Theorem 17. *Let $N \geq 1$, $\tau > 0$, $1 < p < \infty$ and Λ be N –Carleson in every half-plane and relatively dense (for some $r > 0$). Then, the restriction operator R_Λ is an isomorphism from PW_τ^p onto $X_{\tau,r}^p(\Lambda)$ if and only if Λ satisfies $(H_N)_{\tau,p}$.*

Remark 18. We will see in the following that $(H_N)_{\tau,p}$ – (ii) can be replaced by (ii)', which is

- (ii)' There exists a subsequence $\Gamma = \{\gamma_n : n \geq 1\} \subset \Lambda$, still relatively dense, such that, if σ_{γ_n} is the set containing γ_n , the sequence

$$\left(\frac{|S'(\gamma_n)|^p}{\prod_{\substack{\lambda \in \sigma_{\gamma_n} \\ \lambda \neq \gamma_n}} |\gamma_n - \lambda|^p} \right)_{n \geq 1}$$

satisfies the discrete Muckenhoupt condition (\mathfrak{A}_p) .

It is clear that for $N = 1$, $d_1(x) = d(x, \Lambda)$ and $(H_1)_{\tau,p}$ with the Carleson condition and the relative density corresponds exactly to the $(LS)_{\tau,p}$ conditions. The proof of Theorem 17 will be done in Section 5.

Remark 19. The choice of $\epsilon = r$ in our construction ensures that, for every $x \in \mathbb{R}$, $\tau_{n_x} = \sigma'_{n_x}$ and permits us to avoid tedious considerations but the conclusion of Theorem 17 is still true with any choice of $\epsilon > 0$.

We will discuss below the necessity of the N –Carleson condition in Theorem 24. In Theorem 17, the definition of the range of R_Λ definitely depends on the partition of Λ which is possible because of the N –Carleson condition. In Section 6, we will construct a space without the *a priori* assumption that Λ is N –Carleson in every half-plane.

5. PROOFS

5.1. Proof of Proposition 14.

Proof. Let us suppose to the contrary that there exists a real sequence $\{x_j\}_{j \geq 1}$ and a sequence of positive numbers $\{r_j\}_{j \geq 1}$ such that $r_j \rightarrow \infty$, $j \rightarrow \infty$ and

$$B(x_j, r_j) \cap \Lambda = \emptyset.$$

We consider the functions of PW_τ^p defined by

$$f_j(z) := \frac{\sin \tau(z - x_j)}{\tau(z - x_j)}, \quad z \in \mathbb{C}, \quad j \geq 1.$$

Since R_Λ is an isomorphism, we obtain that

$$1 \asymp \|f_j\|_p^p \asymp \|R_\Lambda f_j\|_{X_{\tau,\epsilon}^p(\Lambda)}^p.$$

We will show that $\|R_\Lambda f_j\|_{X_{\tau}^p(\Lambda)}^p \rightarrow 0$, $j \rightarrow \infty$, which implies the required contradiction. From the definition, we have

$$\|R_\Lambda f_j\|_{X_{\tau,\epsilon}^p(\Lambda)}^p = \sum_{n \geq 1} (1 + |\operatorname{Im}(\lambda_{n,0})|) \sum_{k=1}^{|\tau_n|} \left| \tilde{\Delta}_{\tau_n}^{k-1} (f_j e^{\pm i\tau \cdot} (\lambda_n^{(k)})) \right|^p.$$

Using Lemme 5 (see [6, p. 95] for details), we can see that, for every $n \geq 1$ and every $1 \leq k \leq |\tau_n|$,

$$\left| \tilde{\Delta}_{\tau_n}^{k-1} (f_j e^{\pm i\tau \cdot} (\lambda_n^{(k)})) \right|^p \lesssim \frac{1}{|\lambda_{n,0} - x_j|^p},$$

which implies

$$\|R_\Lambda f_j\|_{X_{\tau,\epsilon}^p(\Lambda)}^p \lesssim \sum_{n \geq 1} \frac{1 + |\operatorname{Im}(\lambda_{n,0})|}{|\lambda_{n,0} - x_j|^p}.$$

On the other hand, $p > 1$ and so we can find $\alpha > 0$ such that $p - \alpha > 1$. Recall that $|\lambda_{n,0} - x_j| \geq r_j$ and let us write

$$\|R_\Lambda f_j\|_{X_{\tau,\epsilon}^p(\Lambda)}^p \lesssim \frac{1}{r_j^\alpha} \sum_{n \geq 1} \frac{1 + |\operatorname{Im}(\lambda_{n,0})|}{|\lambda_{n,0} - x_j|^{p-\alpha}}.$$

We split this sum in two parts, writing $\{\lambda_{n,0} : n \geq 1\} = A^+ \cup A^-$, where

$$A^+ \subset (\mathbb{C}^+ \cup \mathbb{R}) \subset \mathbb{C}_{-\frac{1}{2}}^+$$

and

$$A^- \subset \mathbb{C}^- \subset \mathbb{C}_{\frac{1}{2}}^-.$$

Since $r_j \rightarrow \infty$, $j \rightarrow \infty$, we obtain $|\lambda_{n,0} - x_j| \asymp |\lambda_{n,0} - x_j \pm i|$. It follows that the functions

$$g^\pm : z \mapsto \frac{1}{z - x_j \pm i} \in H^{p-\alpha} \left(\mathbb{C}_{\mp \frac{1}{2}}^\pm \right).$$

Now, A^\pm is Carleson in $\mathbb{C}_{\mp \frac{1}{2}}^\pm$, thus

$$\sum_{\lambda \in A^\pm} \frac{1 + |\operatorname{Im}(\lambda)|}{|\lambda - x_j \pm i|^{p-\alpha}} = \sum_{\lambda \in A^\pm} \frac{1 + |\operatorname{Im}(\lambda)|}{|g^\pm(\lambda)|^{p-\alpha}} \lesssim \|g\|_{H^{p-\alpha}(\mathbb{C}_{\mp \frac{1}{2}}^\pm)}^{p-\alpha} \lesssim 1.$$

We finally obtain that

$$\|R_\Lambda f_j\|_{X_{\tau,\epsilon}^p(\Lambda)}^p \lesssim \frac{1}{r_j^\alpha} \rightarrow 0, j \rightarrow \infty,$$

which is the required contradiction and ends the proof. \square

5.2. Proof of Theorem 17. The proof of Theorem 17 follows the main ideas of Lyubarskii and Seip's paper but needs an important technical work to characterize this more general case.

5.2.1. Paley-Wiener Spaces. We will need some well known facts about Paley-Wiener spaces that we recall here. First, we have the Plancherel-Polyà inequality (see e.g. [11] or [18, p. 95]).

Proposition 20. (*Plancherel-Polyà*). Let $f \in PW_\tau^p$ and $a \in \mathbb{R}$. Then,

$$\int_{-\infty}^{+\infty} |f(x + ia)|^p dx \leq e^{\tau p|a|} \|f\|_p^p.$$

It follows that for every $f \in PW_\tau^p$, the function $z \mapsto e^{i\tau z} f(z)$ belongs to H_+^p . It also follows that translation is an isomorphism from PW_τ^p onto itself. The second fact is a pointwise estimate; there exists a constant $C = C(p)$ such that for every $f \in PW_\tau^p$, we have

$$(5.1) \quad |f(z)| \leq C \|f\|_p (1 + |\operatorname{Im}(z)|)^{-\frac{1}{p}} e^{\tau |\operatorname{Im}(z)|}, \quad z \in \mathbb{C}.$$

5.2.2. Necessary conditions. Let us do the construction of subsection 4.1 with $\epsilon = r$ and suppose that R_Λ is an isomorphism between PW_τ^p and $X_{\tau,\epsilon}^p(\Lambda)$. The necessity of $(H_N) - (i)$ can be shown exactly as in [12] and so we do not prove it here. We first show that the condition $(ii)'$ is necessary. Then, with a technical lemma, adapted from [12], we prove that $(ii)'$ implies (ii) .

Since R_Λ is bijective, for each $\lambda \in \Lambda$, there is a unique function $f_\lambda \in PW_\tau^p$ such that

$$f_\lambda(\mu) = \begin{cases} 1, & \text{if } \mu = \lambda \\ 0, & \text{if } \mu \neq \lambda \end{cases}.$$

As in [12], it can be shown that f_λ only vanishes on $\Lambda \setminus \{\lambda\}$ and that f_λ is of exponential type τ (if its type was $\tau' < \tau$ then considering the function $e^{i(\tau-\tau')(-\lambda)} f_\lambda$, we would obtain a contradiction with the injectivity of R_Λ). Moreover, $z \mapsto (z - \lambda) f_\lambda(z)$ is a function of the Cartwright Class \mathcal{C} vanishing exactly on Λ (see e.g. [11] for definition and general results on \mathcal{C}). Hence, since S is also of exponential type τ , $S(z) = c_\lambda (z - \lambda) f_\lambda(z)$, $z \in \mathbb{C}$, or

$$f_\lambda(z) = \frac{S(z)}{S'(\lambda)(z - \lambda)}.$$

For each $n \geq 1$, the holomorphic function

$$g_n : z \mapsto \frac{S(z)}{\prod_{\lambda \in \sigma'_n} (z - \lambda)}$$

does not vanish in Ω_n (see Formula 4.1). Moreover, choosing $\lambda'_{n,0} \in \sigma'_n$,

$$g_n(\lambda_{n,0}) = \frac{S'(\lambda'_{n,0})}{\prod_{\substack{\lambda \in \sigma'_n \\ \lambda \neq \lambda_{n,0}}} (\lambda'_{n,0} - \lambda)}.$$

Hence, it follows from the maximum and the minimum principle that

$$\inf_{\xi \in \partial \Omega_n} \left| \frac{S(\xi)}{\prod_{\lambda \in \sigma'_n} (\xi - \lambda)} \right| \leq \left| \frac{S'(\lambda'_{n,0})}{\prod_{\substack{\lambda \in \sigma'_n \\ \lambda \neq \lambda_{n,0}}} (\lambda'_{n,0} - \lambda)} \right| \leq \sup_{\xi \in \partial \Omega_n} \left| \frac{S(\xi)}{\prod_{\lambda \in \sigma'_n} (\xi - \lambda)} \right|.$$

From the intermediate values theorem, we deduce the existence of a point $\theta_n \in \partial \Omega_n$ such that

$$(5.2) \quad |S(\theta_n)| = \delta \frac{|S'(\lambda'_{n,0})|}{\prod_{\substack{\lambda \in \sigma'_n \\ \lambda \neq \lambda_{n,0}}} |\lambda'_{n,0} - \lambda|} =: \delta \omega_n.$$

Let us consider now a subsequence $\Gamma := (\gamma_n)_{n \geq 1}$ of $\{\lambda'_{n,0} : n \geq 1\}$ which is still relatively dense and such that

$$\inf_{n \geq 1} (\operatorname{Re}(\gamma_{n+1}) - \operatorname{Re}(\gamma_n)) > 0.$$

We define σ_{γ_n} as the set containing γ_n . The sequence $\Theta := (\theta_n)_{\geq 1}$ denotes the previous θ_n , corresponding to γ_n , and for $n \geq 1$, we set

$$\omega_n := \left| \frac{S'(\gamma_n)}{\prod_{\substack{\lambda \in \sigma_{\gamma_n} \\ \lambda \neq \gamma_n}} (\gamma_n - \lambda)} \right|$$

so that

$$|S(\theta_n)| = \delta \omega_n.$$

We show that the discrete Hilbert transform $\mathcal{H}_{\Gamma, \Theta}$ is bounded from $l^p(\omega)$ into itself. Indeed, let $(a_n)_{n \geq 1}$ be a finite sequence of $l^p(\omega)$. Then, the sequence

$$a(\lambda) := \begin{cases} a_n S'(\gamma_n) & , \text{ if } \lambda = \gamma_n \\ 0 & , \text{ if } \lambda \in \Lambda \setminus \Gamma \end{cases}$$

belongs to $X_\tau^p(\Lambda)$ because, if $\gamma_k = \lambda'_{n,0} = \lambda_{n,|\sigma_{\gamma_n}|}$ is chosen as the “last” point of σ'_n ,

$$\tilde{\Delta}_{\sigma_n}^{k-1}(ae^{i\tau \cdot}(\lambda_n^{(k)})) = 0, \quad k < |\sigma_n|$$

and

$$\left| \tilde{\Delta}_{\sigma_n}^{|\sigma_n|-1}(ae^{i\tau \cdot}(\lambda_n^{(|\sigma_n|)})) \right| = \frac{|a_n S'(\lambda'_{n,0})| e^{-\tau |\text{Im}(\lambda'_{n,0})|}}{\prod_{\substack{\lambda \in \sigma'_n \\ \lambda \neq \lambda'_{n,0}}} |\lambda - \lambda'_{n,0}|}.$$

Thus, from (5.2), we obtain, observing that $1 + |\text{Im}(\lambda'_{n,0})|$ and $|e^{i\tau \lambda}|$, $\lambda \in \sigma'_n$, are comparable to a constant since σ'_n is close to \mathbb{R} ,

$$\begin{aligned} \|a\|_{X_\tau^p(\Lambda)}^p &= \sum_n \left(1 + |\text{Im}(\lambda'_{n,0})| \right) \left| \tilde{\Delta}_{\sigma'_n}^{|\sigma'_n|-1}(ae^{i\tau \cdot}(\lambda_n^{(|\sigma'_n|)})) \right|^p \\ (5.3) \quad &\asymp \sum_n \left(\frac{|a_n S'(\lambda'_{n,0})|}{\prod_{\substack{\lambda \in \sigma'_n \\ \lambda \neq \lambda'_{n,0}}} |\lambda - \lambda'_{n,0}|} \right)^p = \sum_n \omega_n^p |a_n|^p. \end{aligned}$$

So, let $f \in PW_\tau^p$ be the (unique) solution of the interpolation problem $f|_\Lambda = a$. Notice that, since R_Λ is an isomorphism onto $X_\tau^p(\Lambda)$, then

$$(5.4) \quad \|f\|_p^p \lesssim \|a\|_{X_\tau^p(\Lambda)}^p.$$

This function is of the form $f(z) = \sum_j a_j \frac{S(z)}{z - \gamma_j}$ and so, with (5.2) we have

$$\sum_n |f(\theta_n)|^p = \sum_n \left| \sum_j a_j \frac{S(\theta_n)}{\theta_n - \gamma_j} \right|^p = \sum_n |S(\theta_n)| \left| \sum_j \frac{a_j}{\theta_n - \gamma_j} \right|^p$$

and, from the construction of Θ , we obtain

$$(5.5) \quad \sum_n |f(\theta_n)|^p = \delta^p \sum_n \omega_n^p \left| \left(\mathcal{H}_{\Gamma, \Theta}((a_j)_{j \geq 1}) \right)_n \right|^p.$$

On the other hand, the Polyà inequality (see [11, Lecture 20]), and the inequalities (5.4) and (5.3) give

$$(5.6) \quad \sum_n |f(\theta_n)|^p \lesssim \|f\|_p^p \lesssim \|a\|_{X_{\tau}^p(\Lambda)}^p \lesssim \sum_n \omega_n^p |a_n|^p.$$

From (5.5) and (5.6), we deduce that $\mathcal{H}_{\Gamma, \Theta}$ is bounded from $l^p(\omega^p)$ into itself. Using a slight modified version of [12, Lemma 1], we can conclude that the weight $(\omega_n^p)_{n \geq 1}$ satisfies the discrete Muckenhoupt condition (\mathfrak{A}_p) .

Remark 21. It follows from the weak density condition $((H_N) - (i))$, the Generalized Carleson condition (3.1) on $(B_{\sigma_{\gamma_n}})_n$ and the growth of the sequence $(\text{Re}(\gamma_n))_n$ that we have $\text{Re}(\gamma_{n+1}) - \text{Re}(\gamma_n) \leq 3\epsilon$. This implies that

$$\delta'_0 \leq |\gamma_n - \gamma_{n+1}| \leq 4\epsilon.$$

Now, in order to prove (iii), we use the following lemma, adapted from [12, Lemma 2].

Lemma 22. *Suppose $x \in \mathbb{R}$ and $\text{Re}(\gamma_n) \leq x \leq \text{Re}(\gamma_{n+1})$. Then, there exists an $\alpha = \alpha(x) \in [0, 1]$ such that*

$$\omega_n^\alpha \omega_{n+1}^{1-\alpha} \asymp \frac{|S(x)|}{d_N(x)},$$

uniformly with respect to $x \in \mathbb{R}$.

Assuming this lemma to hold, (iii) follows from (iii)' and the inequality $t^\alpha s^{1-\alpha} \leq t + s$, $t, s > 0$ and $\alpha \in [0, 1]$ (we still refer to [6] for details).

Proof. For $x \in [\text{Re}(\gamma_n), \text{Re}(\gamma_{n+1})]$, we set $N(x) := \{n : d(\sigma'_n, x) < \epsilon\}$ and

$$\Lambda(x) := \left(\bigcup_{n \in N(x)} \sigma'_n \right) \cup \sigma_{\gamma_n} \cup \sigma_{\gamma_{n+1}}.$$

Notice that σ_{γ_n} and $\sigma_{\gamma_{n+1}}$ may be subsets of $\bigcup_{n \in N(x)} \sigma'_n$. Observe also that since Λ is a finite union of Carleson sequences, we have

$$\sup_{x \in \mathbb{R}} |N(x)| < \infty.$$

For $\alpha \in [0, 1]$, we want to show that $\vartheta \asymp 1$, where

$$\vartheta := \frac{\omega_n^\alpha \omega_{n+1}^{1-\alpha} d_N(x)}{|S(x)|},$$

and $x \notin \Lambda$ (this is not restrictive since the expression extends continuously to Λ). From the definition of S , we have that

$$S'(\lambda) = -\frac{1}{\lambda} \prod_{\substack{\mu \in \Lambda \\ \mu \neq \lambda}} \left(1 - \frac{\lambda}{\mu}\right), \quad \lambda \in \Lambda.$$

In order to not overcharge notation, all infinite products occurring below will be understood as symmetric limits of finite products:

$$\prod_{\lambda \in \Lambda} a(\lambda) = \lim_{R \rightarrow \infty} \prod_{|\lambda| \leq R} a(\lambda).$$

Thus,

$$\vartheta = \left(\frac{\left| \frac{1}{\gamma_n} \prod_{\lambda \in \Lambda \setminus \{\gamma_n\}} \left(1 - \frac{\gamma_n}{\lambda}\right) \right|^\alpha \left| \frac{1}{\gamma_{n+1}} \prod_{\lambda \in \Lambda \setminus \{\gamma_{n+1}\}} \left(1 - \frac{\gamma_{n+1}}{\lambda}\right) \right|^{1-\alpha} d_N(x)}{\prod_{\lambda \in \Lambda} \left(1 - \frac{x}{\lambda}\right) \prod_{\lambda \in \sigma_{\gamma_n} \setminus \{\gamma_n\}} |\lambda - \gamma_n|^\alpha \prod_{\lambda \in \sigma_{\gamma_{n+1}} \setminus \{\gamma_{n+1}\}} |\lambda - \gamma_{n+1}|^{1-\alpha}} \right).$$

For $\lambda \in \Lambda \setminus \{\gamma_n, \gamma_{n+1}\}$,

$$\frac{\left|1 - \frac{\gamma_n}{\lambda}\right|^\alpha \left|1 - \frac{\gamma_{n+1}}{\lambda}\right|^{1-\alpha}}{\left|1 - \frac{x}{\lambda}\right|} = \frac{|\lambda - \gamma_n|^\alpha |\lambda - \gamma_{n+1}|^{1-\alpha}}{|x - \lambda|}.$$

Note also that for the remaining two points γ_n, γ_{n+1} we have:

$$\frac{\left| \frac{1}{\gamma_n} \left(1 - \frac{\gamma_n}{\gamma_{n+1}}\right) \right|^\alpha \left| \frac{1}{\gamma_{n+1}} \left(1 - \frac{\gamma_{n+1}}{\gamma_n}\right) \right|^{1-\alpha}}{\left| \left(1 - \frac{x}{\gamma_n}\right) \left(1 - \frac{x}{\gamma_{n+1}}\right) \right|} = \frac{|\gamma_{n+1} - \gamma_n|^\alpha |\gamma_n - \gamma_{n+1}|^{1-\alpha}}{|\gamma_n - x| |\gamma_{n+1} - x|}.$$

Now, we split ϑ in two products $\vartheta = \Pi_1(x) \cdot \Pi_2(x)$ corresponding essentially to zeros in $\Lambda(x)$ and zeros in $\Lambda \setminus \Lambda(x)$ ($d_N(x)$) appearing

in Π_1):

$$\begin{aligned}\Pi_1(x) &= \frac{\prod_{\lambda \in \Lambda(x) \setminus \{\gamma_n\}} |\lambda - \gamma_n|^\alpha \prod_{\lambda \in \Lambda(x) \setminus \{\gamma_{n+1}\}} |\lambda - \gamma_{n+1}|^{1-\alpha} d_N(x)}{\prod_{\lambda \in \Lambda(x)} |\lambda - x| \prod_{\lambda \in \sigma_{\gamma_n} \setminus \{\gamma_n\}} |\lambda - \gamma_n|^\alpha \prod_{\lambda \in \sigma_{\gamma_{n+1}} \setminus \{\gamma_{n+1}\}} |\lambda - \gamma_{n+1}|^{1-\alpha}} \\ &= \frac{\prod_{\lambda \in \Lambda(x) \setminus \sigma_{\gamma_n}} |\lambda - \gamma_n|^\alpha \prod_{\lambda \in \Lambda(x) \setminus \sigma_{\gamma_{n+1}}} |\lambda - \gamma_{n+1}|^{1-\alpha} d_N(x)}{\prod_{\lambda \in \Lambda(x)} |\lambda - x|}\end{aligned}$$

and

$$\Pi_2(x) := \prod_{\lambda \in \Lambda \setminus \Lambda(x)} \left(\frac{|\lambda - \gamma_n|^\alpha |\lambda - \gamma_{n+1}|^{1-\alpha}}{|\lambda - x|} \right).$$

We can write

$$\begin{aligned}\Pi_1(x) &= \left(\frac{\prod_{\lambda \in \sigma_{\gamma_{n+1}}} |\lambda - \gamma_n|^\alpha \prod_{\lambda \in \sigma_{\gamma_n}} |\lambda - \gamma_{n+1}|^{1-\alpha} d_N(x)}{\prod_{\sigma_{\gamma_n} \cup \sigma_{\gamma_{n+1}}} |x - \lambda|} \right) \\ &\quad \times \left(\prod_{\Lambda(x) \setminus (\sigma_{\gamma_n} \cup \sigma_{\gamma_{n+1}})} \frac{|\lambda - \gamma_n|^\alpha |\lambda - \gamma_{n+1}|^{1-\alpha}}{|x - \lambda|} \right)\end{aligned}$$

and notice that if $\lambda \in \Lambda(x) \setminus (\sigma_{\gamma_n} \cup \sigma_{\gamma_{n+1}})$, then $\lambda \in \sigma'_l$ for a suitable $l \in N(x)$, so that

$$1 \lesssim d(\sigma_{\gamma_n}, \sigma'_l) \leq |\lambda - \gamma_n| \leq 2\rho'_0 + 2\epsilon \lesssim 1$$

and, in view of Remark 21, for $\lambda \in \sigma_{\gamma_n}$ and $\mu \in \sigma_{\gamma_{n+1}}$, we have

$$|\lambda - \gamma_{n+1}| \asymp 1 \text{ and } |\mu - \gamma_n| \asymp 1.$$

These three relations imply that

$$\Pi_1(x) \asymp \frac{d_N(x)}{\prod_{\lambda \in \Lambda(x)} |x - \lambda|}.$$

Now, let n_x be such that $d_N(x) = p_{n_x}(x)$ (we refer to Remark 15). Clearly $n_x \in N(x)$. Note also that for $\lambda \in \sigma'_m$, $m \in N(x)$, we have $|\lambda - x| \leq d(\sigma'_m, x) + \text{diam}(\sigma'_m) \leq \epsilon + \rho'_0$. Hence

$$\frac{1}{(\epsilon + \rho'_0)^{|N(x)|-1}} \leq \frac{d_N(x)}{\prod_{\lambda \in \Lambda(x)} |x - \lambda|} = \frac{1}{\prod_{\lambda \in \Lambda(x) \setminus \sigma_{n_x}} |\lambda - x|} \leq \left(\frac{2}{\delta'_0} \right)^{N \cdot (|N(x)|-1)}$$

and, from the end of Remark 15, we obtain that

$$\Pi_1(x) \asymp 1.$$

The relation

$$\Pi_2(x) \asymp 1$$

is shown exactly in the same way as in [12], using the N -Carleson condition. The lemma is proved. \square

5.2.3. Sufficient conditions. We show the converse of the theorem in two parts; first, the injectivity of R_Λ and then its surjectivity.

Let $f \in PW_\tau^p$ such that $f(\lambda) = 0$, $\lambda \in \Lambda$. We want to show that $f \equiv 0$. Let us introduce $\phi := f/S$. It can be shown that ϕ is an entire function of exponential type 0 (see [6, pp. 96-98] for details). The idea of the proof, given by Lyubarskii and Seip in [12], is to bound ϕ by a constant on the imaginary axis and to use a Phragmen-Lindelöf theorem to obtain that ϕ is a constant. Then, for integrability reasons, the only possible value for the constant will be zero.

We will proceed as follows: since ϕ is analytic, it is bounded on the compact $[-2i\epsilon, 2i\epsilon]$. In order to bound ϕ on $i\mathbb{R} \setminus [-2i\epsilon, 2i\epsilon]$, we will use a lower estimate for S in a certain area of \mathbb{C} . Let us introduce

$$A_n := \{z \in \mathbb{C} : |\operatorname{Im}(z)| \geq 2\epsilon, \rho(\lambda_{n,0}, z) < 2\rho_0 < \epsilon\}, \quad n \in \mathbb{Z}.$$

We begin to show that for $z \in (\mathbb{C}_{2\epsilon}^+ \cup \mathbb{C}_{-2\epsilon}^-) \setminus (\bigcup_n A_n)$,

$$(5.7) \quad |S(z)| \gtrsim e^{\tau|\operatorname{Im}(z)|} (|\operatorname{Im}(z)|)^{\frac{1}{q}} (1 + |z|)^{-1}.$$

Indeed, let us introduce

$$S_1(z) := (S/B_\epsilon)(z),$$

where

$$B_\epsilon(z) := \prod_{\lambda \in \Lambda_\epsilon} \left(c_\lambda \frac{z - \lambda}{z - \bar{\lambda} + 3i\epsilon} \right),$$

is the Blaschke product in $\mathbb{C}_{-\frac{3}{2}\epsilon}^+$ associated to Λ_ϵ , and c_λ is the unimodular normalizing constant which ensures the convergence of the Blaschke product (we do not need the explicit value here). Let $x \in \mathbb{R}$. Observe that for $n \geq 1$ and $\lambda \in \sigma'_{n_x}$, we have

$$|x - \bar{\lambda}| = |x - \lambda| \leq \epsilon + \operatorname{diam}(\sigma'_{n_x}) \leq \epsilon + \rho'_0 \lesssim 1.$$

Hence,

$$|x - \bar{\lambda} + 3i\epsilon| \asymp 1.$$

It follows from these inequalities that

$$\left(\prod_{\lambda \in \sigma'_{n_x}} \left| \frac{x - \lambda}{x - \bar{\lambda} + 3i\epsilon} \right| \right) \asymp d_N(x).$$

Writing

$$|B_\epsilon(x)| = \left(\prod_{\lambda \in \sigma'_{n_x}} \left| \frac{x - \lambda}{x - \bar{\lambda} + 3i\epsilon} \right| \right) \left(\prod_{\lambda \in \Lambda_\epsilon \setminus \sigma'_{n_x}} \left| \frac{x - \lambda}{x - \bar{\lambda} + 3i\epsilon} \right| \right)$$

and using the fact that Λ_ϵ is N -Carleson in $\mathbb{C}_{-\frac{3}{2}\epsilon}^+$, we have then that

$$(5.8) \quad |B_\epsilon(x)| \asymp d_N(x),$$

and so $x \mapsto |S_1(x)|^p$ satisfies (A_p) .

In particular, the function $z \mapsto e^{i\tau z} \frac{S_1(z)}{z+i} = e^{i\tau z} \frac{S(z)}{B_\epsilon(z+i)}$ belongs to H_+^p and the function $z \mapsto e^{i\tau z} S_1(z)$ is a function of \mathcal{N}^+ , the Smirnov Class in the upper half-plane (for definition and general results, see e.g. [15, A.4]). Hence, we can write

$$S_1(z) = e^{-i\tau z} B_1(z) G_1(z), \quad z \in \mathbb{C}^+,$$

where B_1 is the Blaschke product associated to $\Lambda^+ \setminus \Lambda_\epsilon$ and G_1 is an outer function in \mathbb{C}^+ (observe that $e^{i\tau} S_1$ cannot contain any inner singular factor). Thus, $x \mapsto |G_1(x)|^p$ satisfies (A_p) or equivalently, $x \mapsto |G_1(x)|^{-q}$ satisfies (A_q) , with $\frac{1}{p} + \frac{1}{q} = 1$. So, it follows from properties of functions satisfying Muckenhoupt's (A_p) condition, that

$$\phi_{G_1} : z \mapsto \frac{1}{G_1(z)(z+i)} \in H_+^q$$

and, from well known estimates in H_+^q , we get

$$|\phi_{G_1}(z)| \lesssim \frac{1}{(\text{Im}(z))^{\frac{1}{q}}},$$

and so, for $z \in \mathbb{C}^+$,

$$\left| \frac{1}{G_1(z)} \right| \lesssim (1 + |z|) (\text{Im}(z))^{-\frac{1}{q}}.$$

Moreover, because of the N -Carleson condition of $\Lambda^+ \setminus \Lambda_\epsilon$, we have that

$$|B_1(z)| \gtrsim 1, \quad z \in \mathbb{C}^+ \setminus \left(\bigcup_{n \geq 0} A_n \right)$$

and so we do have the lower bound for S_1 stated in (5.7). We notice that $|S(z)| \asymp |S_1(z)|$, $\text{Im}(z) > 2\epsilon$ and so we have the same bound for S in $\mathbb{C}_{2\epsilon}^+$. A similar reasonning gives us the estimate in $\mathbb{C}_{-2\epsilon}^-$.

Using now (5.1) and (5.7), we have for $z \in (\mathbb{C}_{2\epsilon}^+ \cup \mathbb{C}_{-2\epsilon}^-) \setminus (\bigcup_n A_n)$,

$$\begin{aligned} |\phi(z)| &= \left| \frac{f(z)}{S(z)} \right| \lesssim \frac{(1 + |z|)}{e^{\tau|\text{Im}(z)|} |\text{Im}(z)|^{\frac{1}{q}} (1 + |\text{Im}(z)|)^{\frac{1}{p}}} \\ &\asymp \frac{(1 + |z|)}{|\text{Im}(z)|^{\frac{1}{q}} (1 + |\text{Im}(z)|)^{\frac{1}{p}}} =: \psi(z). \end{aligned}$$

We notice then that if $A_n \cap i\mathbb{R} \neq \emptyset$, then

$$A_n \subset S_{\pm} := \left\{ z \in \mathbb{C}^{\pm} : \left| \frac{\text{Im}(z)}{\text{Re}(z)} \right| < \eta \right\},$$

where η is a suitable constant. Note that S_{\pm} are Stolz angles in \mathbb{C}^{\pm} at $x = 0$. Since A_n is far from \mathbb{R} and has uniformly bounded pseudohyperbolic diameter, every A_n hitting the imaginary axis will be in the Stolz angle S_+ or S_- . Obviously, there is some $M > 0$ such that for every $z \in \mathbb{C}_{\pm 2\epsilon}^{\pm} \cap S_{\pm}$, we have

$$|\psi(z)| \leq M.$$

In particular, $|\phi(z)| \leq M$ for $z \in \partial A_n$ and by the maximum principle,

$$|\phi(iy)| \leq M \text{ for } iy \in A_n \cap i\mathbb{R}.$$

Hence, ϕ is uniformly bounded on $i\mathbb{R}$ and it follows, by a Phragmen-Lindelöf principle that $\phi \equiv K$, and $f = KS$. Let us now show that $K = 0$. Because $x \mapsto |S_1(x)|^p$ satisfies (A_p) we have

$$\int |S_1(x)|^p = \infty$$

and, applying the Plancherel-Polyà inequality, we also have

$$\int |S_1(x + 2i\epsilon)|^p = \infty$$

but $|S(x + 2i\epsilon)| \asymp |S_1(x + 2i\epsilon)|$, so

$$\int |S_1(x + 2i\epsilon)|^p = \infty.$$

We apply again the Plancherel-Polyà inequality to obtain

$$\int |S(x)|^p = \infty.$$

From the fact that $f \in PW_{\tau}^p$, we have by definition that $f \in L^p$ and since $f = \phi S = KS$, the only possibility is $K = 0$ and so $f \equiv 0$, which

ends the proof of the injectivity of R_Λ . Now, we can show the last part of the proof.

Let us consider a finite sequence $a = (a(\lambda))_{\lambda \in \Lambda}$ and the solution of the interpolation problem $f(\lambda) = a(\lambda)$, $\lambda \in \Lambda$, given by

$$f(z) = \sum_{\lambda \in \Lambda} a(\lambda) \frac{S(z)}{S'(\lambda)(z - \lambda)}.$$

Since the sum is finite, f is an entire function of type at most τ . We want to split this sum according to the localization of the points of Λ . More precisely, we recall that we have the decomposition $\Lambda = \bigcup_{n \in \mathbb{Z}} \tau_n$ and we have already introduced

$$N_+ = \{n \in \mathbb{Z} : \tau_n \cap (\mathbb{C}^+ \cup \mathbb{R}) \neq \emptyset\} \text{ and } N_- = \mathbb{Z} \setminus N_+.$$

We set

$$\Lambda_+ := \bigcup_{n \in N_+} \tau_n \text{ and } \Lambda_- := \bigcup_{n \in N_-} \tau_n = \Lambda \setminus \Lambda_+.$$

(Observe that since $\text{diam}(\tau_n) < \frac{\epsilon}{2}$, we have $\Lambda_+ \subset \mathbb{C}_{-\frac{\epsilon}{2}}^+$). Now, we can write $f = f^+ + f^-$, with

$$f^\pm(z) := \sum_{\lambda \in \Lambda_\pm} a(\lambda) \frac{S(z)}{S'(\lambda)(z - \lambda)} = \sum_{n \in N_\pm} \sum_{\lambda \in \tau_n} a(\lambda) \frac{S(z)}{S'(\lambda)(z - \lambda)}.$$

We want to estimate, separately,

$$(5.9) \quad \inf \left\{ \|f^\pm - g\|_p : g \in PW_\tau^p, g|_\Lambda = 0 \right\}.$$

Here we will only consider f^+ , the method is the same for f^- . In the following, β will be the Blaschke product associated to $\Lambda_{-\epsilon}^+ := \Lambda \cap \mathbb{C}_{-\epsilon}^+$

$$\beta(z) = \prod_{\lambda \in \Lambda_{-\epsilon}^+} \left(c_\lambda \frac{z - \lambda}{z - \bar{\lambda} + 2i\epsilon} \right), \quad z \in \mathbb{C}_{-\epsilon}^+,$$

where again c_λ is a suitable normalizing factor. For $z \in \mathbb{C}_{-\epsilon}^+$, we write $S(z) = e^{-i\tau z} \beta(z) G(z)$. Observe that $\beta(0) = \prod_{\lambda \in \Lambda} c_\lambda \frac{\lambda}{\lambda - 2i\epsilon}$ (recall that we have assumed $0 \notin \Lambda$). Thus, we can write

$$\begin{aligned} G(z) &= e^{i\tau z} S(z) \beta(z)^{-1} \\ &= e^{i\tau z} \prod_{\lambda \in \Lambda} \left(\frac{\lambda - z}{\lambda} \right) \prod_{\lambda \in \Lambda_{-\epsilon}^+} \left(c_\lambda \frac{z - \bar{\lambda} + 2i\epsilon}{z - \lambda} \right) \\ &= \beta(0)^{-1} e^{i\tau z} \prod_{\tilde{\lambda} \in \tilde{\Lambda}} \left(1 - \frac{z}{\tilde{\lambda}} \right), \end{aligned}$$

with $\tilde{\Lambda} := (\Lambda \setminus \Lambda_{-\epsilon}^+) \cup (\overline{\Lambda_{-\epsilon}^+} - 2i\epsilon) \subset \mathbb{C}_{-\epsilon}^-$. The function G is outer in $\mathbb{C}_{-\epsilon}^+$. As in (5.8), we obtain $|\beta(x)| \asymp d_N(x)$. In particular, we have $|G(x)|^p \in (A_p)$. Let then be η such that $\frac{\epsilon}{2} < \eta < \epsilon$. Since $\tilde{\Lambda}$ is the union (not necessarily disjoint) of two N -Carleson sequences in $\mathbb{C}_{-\epsilon}^-$, and in particular

$$\operatorname{Im}(\tilde{\lambda} + i\eta) \leq \eta - \epsilon < 0, \quad \tilde{\lambda} \in \tilde{\Lambda},$$

(which implies in particular that every real x is far from $\tilde{\Lambda}$), we obtain that

$$|G(x - i\eta)| = e^{\tau\eta} |G(x)| \left(\prod_{\tilde{\lambda} \in \tilde{\Lambda}} \left| \frac{x - \tilde{\lambda} - i\eta}{x - \tilde{\lambda}} \right| \right) \asymp |G(x)|.$$

So $x \mapsto |G(x - i\eta)|^p$ also satisfies the Muckenhoupt condition (A_p) . According to the Plancherel-Polyà inequality, it is possible to estimate (5.9) on the axis $\{\operatorname{Im}(z) = -\eta\}$.

By duality arguments (see [19, p. 576] or [6, p. 94]), we need to estimate

$$\sup_{\substack{h \in H^q(\mathbb{C}_{-\eta}^+) \\ \|h\|_q=1}} N(h),$$

with

$$\begin{aligned} N(h) : &= \left| \sum_{\lambda \in \Lambda^+} \frac{a(\lambda)}{S'(\lambda)} \int \frac{G(x - i\eta)h(x - i\eta)}{x - i\eta - \lambda} dx \right| \\ &= \left| \sum_{\lambda \in \Lambda^+} \frac{a(\lambda)}{S'(\lambda)} \mathcal{H}(\tilde{G}\tilde{h})(\lambda + i\eta) \right| \end{aligned}$$

where $z \mapsto \tilde{G}(z) = G(z - i\eta)$ is an outer function in \mathbb{C}^+ and the function $z \mapsto \tilde{h}(z) = h(z - i\eta)$ belongs to H_+^q . In order to compute $S'(\lambda)$, let us recall that

$$S(z) = e^{-i\tau z} \beta(z) G(z), \quad z \in \mathbb{C}_{-\eta}^+.$$

For $\lambda \in \tau_n$, $n \in N_+$, we have

$$S'(\lambda) = c_\lambda \frac{e^{-i\tau\lambda}}{\lambda - \bar{\lambda} + 2i\epsilon} G(\lambda) \frac{\beta}{b_\lambda^\epsilon}(\lambda),$$

where $b_\lambda^\epsilon(z) = c_\lambda \frac{z - \lambda}{z - \bar{\lambda} + 2i\epsilon}$. Using that $G(\lambda) = \tilde{G}(\lambda + i\eta)$, and setting

$$\psi := \frac{\mathcal{H}(\tilde{G}\tilde{h})}{\tilde{G}} \text{ and } \alpha(\lambda) := a(\lambda) e^{i\tau\lambda}, \quad \lambda \in \Lambda^+,$$

where we recall that \mathcal{H} denotes the Hilbert transform (see 1.2 for definition) the expression becomes

$$N(h) = \left| \sum_{n \in N_+} \sum_{\lambda \in \tau_n} \frac{\alpha(\lambda)\psi(\lambda + i\eta)}{\prod_{\mu \neq \lambda} b_\mu^\epsilon(\lambda)} (\lambda - \bar{\lambda} + 2i\epsilon) \right|.$$

Writing

$$N_+ = N_\epsilon \cup N_\infty, \text{ with } N_\epsilon := \{n \in N_+ : \tau_n \cap \{|\operatorname{Im}(z)| < \epsilon\} \neq \emptyset\},$$

we set, with the help of the functions of Lemma 6,

$$\begin{aligned} P_{\tau_n, \alpha}(z) &:= \sum_{k=1}^{|\tau_n|} \Delta_{\tau_n}^{k-1}(\alpha(\lambda_n^{(k)})) \prod_{l=1}^{k-1} b_{\lambda_{n,l}}(z), \quad n \in N_\infty, \\ P_{\tau_n, \alpha}(z) &:= \sum_{k=1}^{|\tau_n|} \square_{\tau_n}^{k-1}(\alpha(\lambda_n^{(k)})) \prod_{l=1}^{k-1} (z - \lambda_{n,l}), \quad n \in N_\epsilon \end{aligned}$$

and setting $\tilde{\tau}_n := \tau_n + i\eta$

$$Q_{\tilde{\tau}_n, \psi}(z) := \sum_{k=1}^{|\tilde{\tau}_n|} \Delta_{\tilde{\tau}_n}^{k-1}(\psi(\lambda_{n,1} + i\eta, \dots, \lambda_{n,k} + i\eta)) \prod_{l=1}^{k-1} b_{\lambda_{n,l} + i\eta}(z).$$

We notice that

$$N(h) = \left| \sum_{n \in N_+} \sum_{\lambda \in \tau_n} \frac{P_{\tau_n, \alpha}(\lambda) Q_{\tilde{\tau}_n, \psi}(\lambda + i\eta)}{\prod_{\mu \neq \lambda} b_\mu^\epsilon(\lambda)} (\lambda - \bar{\lambda} + 2i\epsilon) \right|.$$

Recall now that $\tau_n \subset R_n$, where $(R_n)_n$ are the disjoint rectangles (constructed here in the half-plane $\mathbb{C}_{-\eta}^+$ so that in particular satisfying $d(\partial R_n, \mathbb{R} - i\eta) \asymp l_n \asymp L_n$) introduced in Remark 12. (Note also that here we have that $\Lambda_+ \subset \mathbb{C}_{-\frac{\epsilon}{2}}^+$ and in particular, Λ_+ is far from $\mathbb{R} - i\eta$). Then, if $\Gamma_n := \partial R_n$, the function

$$z \mapsto h_n(z) := \frac{P_{\tau_n, \alpha}(z) Q_{\tilde{\tau}_n, \psi}(z + i\eta)}{\beta(z)}$$

is a meromorphic function in $\overset{\circ}{R}_n$ with simple poles at $\lambda \in \tau_n$. Thus, the residue theorem implies that

$$\int_{\Gamma_n} h_n(z) dz = 2i\pi \sum_{\lambda \in \tau_n} \operatorname{Res}(h_n, \lambda)$$

and

$$\text{Res}(h_n, \lambda) = P_{\tau_n, \alpha}(\lambda) Q_{\tilde{\tau}_n, \psi}(\lambda + i\eta) \left(\frac{\beta}{b_\lambda^\epsilon}(\lambda) \right)^{-1} \cdot (\lambda - \bar{\lambda} + 2i\epsilon).$$

It follows that

$$N(h) = \left| \frac{1}{2i\pi} \sum_{n \in N_+} \int_{\Gamma_n} \frac{P_{\tau_n, \alpha}(z) Q_{\tilde{\tau}_n, \psi}(z + i\eta)}{\beta} dz \right|.$$

Obviously $|b_{\lambda_n, l}(z)| \leq 1$. Observe also that by condition (3.5) of Remark 12 for $z \in \Gamma_n$, $n \in N_\epsilon$, we have that $|z - \lambda_{n,l}|$ is bounded by a fixed constant. Hence for every $n \in N_+$,

$$|P_{\tau_n, \alpha}| \lesssim \sum_{k=1}^{|\tau_n|} \left| \tilde{\Delta}_{\tau_n}^{k-1}(\alpha(\lambda_n^{(k)})) \right|.$$

Also

$$|Q_{\tilde{\tau}_n, \psi}| \lesssim \sum_{k=1}^{|\tilde{\tau}_n|} \left| \Delta_{\tilde{\tau}_n}^{k-1}(\psi(\lambda_{n,1} + i\eta, \dots, \lambda_{n,k} + i\eta)) \right|,$$

and we obtain that

$$N(h) \lesssim \sum_{n \in N_+} \left[\left(\int_{\Gamma_n} \left| \frac{dz}{\beta(z)} \right| \right) \left(\sum_{k=1}^{|\tau_n|} \left| \tilde{\Delta}_{\tau_n}^{k-1}(\alpha) \right| \right) \left(\sum_{l=1}^{|\tilde{\tau}_n|} \left| \Delta_{\tilde{\tau}_n}^{l-1}(\psi) \right| \right) \right].$$

For $z \in \Gamma_n$, we see that

$$\begin{aligned} |\beta(z)| &= \left(\prod_{\lambda \in \Lambda_+ \setminus \tau_n} \left| \frac{z - \lambda}{z - \bar{\lambda} + 2i\epsilon} \right| \right) \cdot \left(\prod_{\lambda \in \tau_n} \left| \frac{z - \lambda}{z - \bar{\lambda} + 2i\epsilon} \right| \right) \\ &=: \Pi_1(z) \cdot \Pi_2(z). \end{aligned}$$

Since Λ_+ is N -Carleson in $\mathbb{C}_{-\epsilon}^+$, it follows from the fact that R_n is “far” from τ_k , $k \neq n$ that

$$\Pi_1(z) \asymp 1$$

and from the fact that R_n is “far” from τ_n that

$$\Pi_2(z) \asymp 1.$$

Hence, choosing arbitrarily $\lambda_{n,0} \in \tau_n$, the construction of R_n gives

$$\int_{\Gamma_n} \left| \frac{dz}{\beta(z)} \right| \lesssim \int_{\Gamma_n} |dz| \lesssim \text{Im}(\lambda_{n,0}) + \eta \lesssim 1 + |\text{Im}(\lambda_{n,0})|.$$

Applying Hölder's inequality, we obtain

$$\begin{aligned} N(h) &\lesssim \left(\sum_{n \in N_+} (1 + \operatorname{Im}(\lambda_{n,0})) \sum_{k=1}^{|\tau_n|} \left| \tilde{\Delta}_{\tau_n}^{k-1} (e^{i\tau} \cdot a) \right|^p \right)^{\frac{1}{p}} \\ &\quad \times \left(\sum_{n \in N_+} \operatorname{Im}(\lambda_{n,0} + i\eta) \sum_{k=1}^{|\tilde{\tau}_n|} \left| \Delta_{\tilde{\tau}_n}^{k-1} (\psi) \right|^q \right)^{\frac{1}{q}}. \end{aligned}$$

Now, notice that by the Muckenhoupt condition on $|\tilde{G}|^{-q}$ and thus the boundedness of \mathcal{H} on

$$H_+^q \left(\left| \frac{1}{\tilde{G}} \right|^q \right) := \left\{ f \in \mathcal{N}^+ : f|_{\mathbb{R}} \in L^q \left(\left| \frac{1}{\tilde{G}} \right|^q \right) \right\},$$

(\mathcal{N}^+ denotes the Smirnov class) we get that $\psi \in H_+^q$ and $\|\psi\|_q \lesssim \|\tilde{h}\|_{H_+^q} = 1$. But, since

$$\bigcup_{n \in N_+} \tilde{\tau}_n = \Lambda^+ + i\eta$$

is in fact N -Carleson in $\mathbb{C}_{\eta - \frac{\epsilon}{2}}^+ \subset \mathbb{C}^+$ and $\psi \in H_+^q$, Theorem 13 implies that

$$\left(\sum_{n \in N_+} \operatorname{Im}(\lambda_{n,0} + i\eta) \sum_{k=1}^{|\tilde{\tau}_n|} \left| \Delta_{\tilde{\tau}_n}^{k-1} (\psi) \right|^q \right)^{\frac{1}{q}} \lesssim \|\psi\|_{H_+^q} \lesssim \|\tilde{h}\|_{H_+^q} = 1.$$

Finally, we obtain

$$N(h) \lesssim \left(\sum_{n \in N_+} (1 + \operatorname{Im}(\lambda_{n,0})) \sum_{k=1}^{|\tau_n|} \left| \tilde{\Delta}_{\tau_n}^{k-1} (e^{i\tau} \cdot a) \right|^p \right)^{\frac{1}{p}} = \|a\|_{X_{\tau,\epsilon}^p(\Lambda)},$$

which ends the proof.

6. ABOUT THE N -CARLESON CONDITION

It is clear that the definition of $X_{\tau,\epsilon}^p(\Lambda)$ depends on the N -Carleson hypothesis, and more precisely for the construction of the groups τ_n . In this last section, we show that in a certain way, the N -Carleson condition is necessary.

It will be convenient to introduce the distance function

$$\delta(z, \xi) := \frac{|z - \xi|}{1 + |z - \bar{\xi}|}, \quad z, \xi \in \mathbb{C},$$

which expresses that locally we deal with Euclidian geometry close to the real axis and pseudohyperbolic geometry far away from the real axis (see *e.g.* [17, page 715]). Let $\Lambda = \{\lambda_n\}_{n \geq 1}$ be a sequence of complex numbers. Let $N \geq 1$ be an integer and $\eta \in (0, \frac{1}{2})$. For $\lambda \in \Lambda$, we define

$$D_{\lambda, \eta} := \{z \in \mathbb{C} : \delta(\lambda, z) < \eta\},$$

$$N_\lambda := \{\mu_{\lambda, i} : 1 \leq i \leq N\} \subset \Lambda$$

as the set of N closest neighbors of λ (including in particular λ) with respect to the distance δ . Then we set

$$\sigma_\lambda := D_{\lambda, \eta} \cap N_\lambda, \quad n_\lambda := |\sigma_\lambda| \leq N.$$

Note that the set N_λ , and consequently σ_λ , is not unique. It is now natural to introduce the space (for $1 < p < \infty$)

$$X_\tau^p(\Lambda, N) := \left\{ a = (a(\lambda))_{\lambda \in \Lambda} : \|a\|_{X_\tau^p(\Lambda, N)} < \infty \right\},$$

where

$$\|a\|_{X_\tau^p(\Lambda, N)}^p := \sum_{\lambda \in \Lambda} (1 + |\operatorname{Im}(\lambda)|) \sum_{k=1}^{n_\lambda} \left| \tilde{\Delta}_{\sigma_\lambda}^{k-1} (ae^{\pm i\tau \cdot} (\mu^{(k)})) \right|^p$$

with

$$\tilde{\Delta}_{\sigma_\lambda} = \begin{cases} \Delta_{\sigma_\lambda}, & \text{if } \sigma_\lambda \cap \{z \in \mathbb{C} : |\operatorname{Im}(z)| < 1\} = \emptyset \\ \square_{\sigma_\lambda}, & \text{if not} \end{cases}$$

and

$$e^{\pm i\tau \mu} = \begin{cases} e^{i\tau \mu}, & \text{if } \mu \in \sigma_\lambda \text{ and } \sigma_\lambda \cap \{z \in \mathbb{C} : \operatorname{Im}(z) \geq 0\} \neq \emptyset \\ e^{-i\tau \mu}, & \text{otherwise} \end{cases}.$$

Remark 23. It can be shown that if $\Lambda \cap \mathbb{C}_a^\pm$ is N -Carleson in the corresponding half-plane, for each $a \in \mathbb{R}$, then this norm is equivalent to the previously norm $\|\cdot\|_{X_{\tau, \epsilon}^p(\Lambda)}$ (for every $\epsilon > 0$) defined in the above section. For the proof, we refer to [7, pp. 36-38].

The result is the following one.

Theorem 24. *If R_Λ is an isomorphism from PW_τ^p onto $X_\tau^p(\Lambda, N)$, then for every $a \in \mathbb{R}$, $\Lambda \cap \mathbb{C}_a^\pm$ is N' -Carleson in the corresponding half-plane, with $N' \leq N$.*

The proof is in two parts. We begin by showing that if R_Λ is such an isomorphism, then Λ_a^\pm is N' -Carleson for some $N' \in \mathbb{N}$. This only requires the boundedness of R_Λ . We first notice that by the Plancherel-Polyà theorem (Proposition 20) the map

$$\begin{aligned} \tau_a : PW_\tau^p &\rightarrow PW_\tau^p \\ f &\mapsto f(\cdot + i(1 + |a|)) \end{aligned}$$

is an isomorphism and so $\tilde{R}_\Lambda := R_\Lambda \circ \tau_a$ is still an isomorphism. Obviously, $\tilde{R}_\Lambda = R_{\tilde{\Lambda}}$, where

$$\tilde{\Lambda} := \Lambda + i(1 + |a|).$$

Note that for $\lambda \in \Lambda_a^+$, with the notations of Lemma 6,

$$|a_\lambda|^p e^{-p\text{Im}(\lambda)} = |P_{\sigma_\lambda, e^{\pm i\tau} \cdot a}| \leq \sum_{k=1}^{n_\lambda} \left| \tilde{\Delta}_{\sigma_\lambda}^{k-1} (ae^{\pm i\tau} (\mu^{(k)})) \right|^p$$

and so $X_\tau^p(\tilde{\Lambda}, N)$ injects into $l^p\left(\left(1 + |\text{Im}(\tilde{\lambda})|\right) e^{-p|\text{Im}(\tilde{\lambda})|}\right)$ so that

$$R_{\tilde{\Lambda}} : PW_\tau^p \rightarrow l^p\left(\left(1 + |\text{Im}(\tilde{\lambda})|\right) e^{-p|\text{Im}(\tilde{\lambda})|}\right)$$

is bounded. We set $\tilde{\Lambda}_a^+ := \Lambda_a^+ + i(1 + |a|)$ and reintroduce the inner function $I_\tau(z) = \exp(2i\tau z)$. We have mentioned in the beginning of the paper that PW_τ^p is isomorphic to $K_{I_\tau}^p$, so

$$R_{\tilde{\Lambda}_a^+}^{I_\tau} := R_{\tilde{\Lambda}_a^+} \Big| K_{I_\tau}^p : K_{I_\tau}^p \rightarrow L^p(\mu_{\tilde{\Lambda}_a^+})$$

is bounded, where

$$\mu_{\tilde{\Lambda}_a^+} := \sum_{\tilde{\lambda} \in \tilde{\Lambda}_a^+} \text{Im}(\tilde{\lambda}) \delta_{\tilde{\lambda}}.$$

In order to show that Λ_a^\pm is N' -Carleson, it is sufficient to show that $\mu_{\tilde{\Lambda}_a^+}$ is a Carleson measure for H_+^p . Since in particular $\tilde{\Lambda}_a^+ \subset \mathbb{C}_1^+$, it is possible to find $\epsilon \in (0, 1)$ such that

$$\tilde{\Lambda}_a^+ \subset L(I^\tau, \epsilon) := \{z \in \mathbb{C}^+ : |I^\tau(z)| < \epsilon\}.$$

Now, from a result of Treil and Volberg (see [20] or [1]), the boundedness of $R_{\tilde{\Lambda}_a^+}^{I_\tau}$ implies that

$$(6.1) \quad \sup_I \frac{\mu_{\tilde{\Lambda}_a^+}(\omega_I)}{m(I)} < \infty,$$

where the supremum is taken over all the intervals of finite length such that the Carleson window ω_I constructed on I satisfies

$$\omega_I \cap L(I^\tau, \epsilon) \neq \emptyset.$$

Observe that $L(I^\tau, \epsilon)$ is in the upper half plane \mathbb{C}_b^+ , $b = \log(1/\epsilon)$, so that if the length of the Carleson window is less than b , then we have $\omega_I \cap L(I^\tau, \epsilon) = \emptyset$. Hence, $\omega_I \cap \tilde{\Lambda}_a^+ = \emptyset$ and so $\mu_{\tilde{\Lambda}_a^+}(\omega_I) = 0$. It follows that (6.1) is true for all finite length intervals I , which is equivalent to

the fact that $\mu_{\tilde{\Lambda}_a^+}$ is a Carleson measure or also that $\tilde{\Lambda}_a^+$ is N' -Carleson and hence Λ_a^+ in the corresponding half-plane. Considering the map

$$\begin{aligned} s : \quad PW_\tau^p &\rightarrow PW_\tau^p \\ f &\mapsto f(-\cdot) \end{aligned}$$

which is also an isomorphism, we will also have the result for Λ_a^- .

Now, we want to prove that $N' \leq N$. In the following, if Λ_a^+ is $(N+k)$ -Carleson, we write

$$\Lambda_a^+ = \bigcup_{n \geq 1} \tau_n^k,$$

where the groups τ_n^k come from the Generalized Carleson condition, and so it is possible to assume that

$$\text{diam}_\delta(\tau_n^k) < \frac{\eta}{4}$$

(which in particular implies that $\tau_n^k \subset D_{\lambda, \eta}$) and

$$\gamma := \inf_{n \neq m} \delta(\tau_n^k, \tau_m^k) > 0.$$

We need the following lemma and its corollary. For technical reasons, let us assume (without loss of generality) that $\Lambda_a^+ \subset \mathbb{C}_1^+$ so that we can deal with the pseudohyperbolic metric and the corresponding divided differences.

Lemma 25. *If R_Λ is an isomorphism from PW_τ^p onto $X_\tau^p(\Lambda, N)$ and Λ_a^+ is $(N+k+1)$ -Carleson, $k \geq 0$, then it is possible to find $\vartheta > 0$ such that every τ_n^{k+1} with $|\tau_n^{k+1}| = N+k+1$ satisfies $\text{diam}_\rho(\tau_n^{k+1}) > \vartheta$.*

Proof. Let us suppose to the contrary that we can find a subsequence $(\tilde{\tau}_j)$ of (τ_n^{k+1}) such that $|\tilde{\tau}_j| = N+k+1$ and $\text{diam}_\rho(\tilde{\tau}_j) \rightarrow 0$, $j \rightarrow \infty$. We set $\tilde{\tau}_j = \{\lambda_i^j : i = 0, \dots, N+k\}$. Let us now introduce the sequence $a^j = (a^j(\lambda))_{\lambda \in \Lambda}$ defined by

$$a^j(\lambda) := 0, \quad \lambda \neq \lambda_{N+k}^j,$$

and

$$a^j(\lambda_{N+k}^j) := e^{\tau \text{Im}(\lambda_{N+k}^j)} \text{Im}(\lambda_{N+k}^j)^{-\frac{1}{p}} \frac{\prod_{i \neq N+k} |b_{\lambda_i^j}(\lambda_{N+k}^j)|}{\max_{i \neq N+k} |b_{\lambda_i^j}(\lambda_{N+k}^j)|}.$$

Let

$$M_j := \{\lambda \in \Lambda_a^+ : \lambda_{N+k}^j \in \sigma_\lambda\}$$

the set of the points of Λ_a^+ close to λ_{N+k}^j . Since $\text{diam}_\rho(\tilde{\tau}_j) < \frac{\eta}{4}$ and $\lambda_{N+k}^j \in \tilde{\tau}_j$ we have for every $\lambda \in \tilde{\tau}_j$ that $\lambda_{N+k}^j \in \sigma_\lambda$, i.e. $\tilde{\tau}_j \subset M_j$. So, let $B_j := M_j \setminus \tilde{\tau}_j$. Also, since Λ_a^+ is $(N + k + 1)$ -Carleson,

$$\sup_j |\Lambda_a^+ \cap D_{\lambda_{N+k}^j}| < \infty,$$

which implies that

$$\sup_j |M_j| < \infty.$$

By construction,

$$\|a^j\|_{X_\tau^p(\Lambda, N)}^p = \sum_{\lambda \in M_j} (1 + \text{Im}(\lambda)) \sum_{l=1}^{n_\lambda} |\Delta_{\sigma_\lambda}^{l-1}(a^j e^{i\tau \cdot}(\mu^{(l)}))|^p.$$

(Observe that we only consider in the sum the points containing λ_{N+k}^j in their neighborhood.)

We have to evaluate this expression. Take $\lambda \in M_j$. We recall that $n_\lambda = |\sigma_\lambda|$. Note also that for every $1 \leq l \leq n_\lambda$, the divided difference

$$|\Delta_{\sigma_\lambda}^{l-1}(a^j e^{\pm i\tau \cdot}(\lambda^{(l)}))|$$

will be equal either to 0 or to

$$\left| a^j(\lambda_{N+k}^j) e^{\pm i\tau \lambda_{N+k}^j} \prod_{m \in \omega_l} b_{\lambda_m^j}(\lambda_{N+k}^j) \right|,$$

where $\omega_l \subset \sigma_\lambda$ contains $l - 1$ points. Now, $\omega_l = \omega_{l,1} \cup \omega_{l,2}$ where $\omega_{l,1} = \sigma_\lambda \cap \tilde{\tau}_j$ and $\omega_{l,2}$ are the other points. Note that ω_l cannot contain λ_{N+k}^j . By assumption, for $\mu \in \omega_{l,2}$, $|b_\mu(\lambda_{N+k}^j)| \geq \gamma$. Hence,

$$\begin{aligned} & \sum_{l=1}^{n_\lambda} |\Delta_{\sigma_\lambda}^{l-1}(a^j e^{\pm i\tau \cdot}(\lambda^{(l)}))|^p \\ & \leq \sum_{l=1}^{n_\lambda} \frac{\prod_{i \neq N+k} |b_{\lambda_i^j}(\lambda_{N+k}^j)|^p}{\max_{i \neq N+k} |b_{\lambda_i^j}(\lambda_{N+k}^j)|^p} \cdot \frac{1}{\text{Im}(\lambda_{N+k}^j) \prod_{\mu \in \omega_l} |b_\mu(\lambda_{N+k}^j)|^p} \\ & \leq \sum_{l=1}^{n_\lambda} \frac{1}{\gamma^{p|\omega_{l,2}|}} \frac{\prod_{\xi \in \Omega_l} |b_\xi(\lambda_{N+k}^j)|^p}{\max_{i \neq N+k} |b_{\lambda_i^j}(\lambda_{N+k}^j)|^p} \cdot \frac{1}{\text{Im}(\lambda_{N+k}^j)} \\ & \lesssim \frac{N}{\text{Im}(\lambda_{N+k}^j)}, \end{aligned}$$

where $\Omega_l = \{\lambda_i^j : i = 0, \dots, N+k-1\} \setminus \omega_{l,1}$ are subsets of $\tilde{\tau}_j$. The last of the above inequalities comes from the observation that Ω_l contains at least:

$$N + k - |\omega_{l,1}| \geq N + k - (n_\lambda - 1) \geq N + k - (N - 1) = k + 1 \geq 1$$

points. We deduce that $a^j \in X_\tau^p(\Lambda, N)$ and that its norm is uniformly bounded. Now, since R_Λ is onto, there is $f^j \in PW_\tau^p$ such that $f^j|_\Lambda = a^j$ and

$$\|f^j\|_{PW_\tau^p} \lesssim \|a^j\|_{X_\tau^p(\Lambda, N)} \lesssim 1.$$

Setting $\tilde{f}^j := e^{i\tau_j} f^j$, it follows from the Plancherel-Polyà inequality that $\tilde{f}^j \in H_+^p$ and since Λ_a^+ is $(N+k+1)$ -Carleson in \mathbb{C}^+ , Theorem (13) implies in particular that

$$\text{Im}(\lambda_{N+K}^j) \left| \Delta_{\tilde{\tau}_j}^{N+k} \left(\tilde{f}^j \left((\lambda^j)^{(N+k+1)} \right) \right) \right|^p \lesssim \|f^j\| \lesssim 1.$$

But by construction, we have

$$\text{Im}(\lambda_{N+K}^j) \left| \Delta_{\tilde{\tau}_j}^{N+k} \left(\tilde{f}^j \left((\lambda^j)^{(N+k+1)} \right) \right) \right|^p = \frac{1}{\max_{i \neq N+k} \rho(\lambda_{N+k}^j, \lambda_i^j)}$$

which tends to ∞ , $j \rightarrow \infty$ because $\text{diam}_\rho \tilde{\tau}_j$ tends to 0, $j \rightarrow \infty$, which gives the required contradiction. \square

The following corollary to the previous lemma allows us to end the proof of our theorem.

Corollary 26. *If R_Λ is an isomorphism from PW_τ^p onto $X_\tau^p(\Lambda, N)$ and Λ_a^+ is $(N+k+1)$ -Carleson, $k \geq 0$, then Λ_a^+ is $(N+k)$ -Carleson.*

Proof. We write $\Lambda_a^+ = \bigcup_{n \geq 1} \tau_n^{k+1}$ with $|\tau_n^{k+1}| \leq N+k+1$. Let us suppose that there are infinitely many n for which we have $|\tau_n^{k+1}| = N+k+1$ and let Z be the set of such n . Because of the previous lemma, we can find $\vartheta > 0$ such that $\text{diam}_\rho(\tau_n^{k+1}) > \vartheta$ for $n \in Z$. Then, for every $n \in Z$, it is possible to write $\tau_n^{k+1} = \{\lambda_i^n : i = 1, \dots, N+k+1\}$ such that

$$\rho(\lambda_i, \lambda_{N+k+1}^n) \geq \frac{\vartheta}{2(N+k)}, \quad i = 1, \dots, N+k.$$

It follows that

$$\Lambda_a^+ = \bigcup_{n \notin Z} \tau_n^{k+1} \cup \left(\bigcup_{n \in Z} \tau_n^{k+1} \setminus \{\lambda_{N+k+1}^n\} \right) \cup \left(\bigcup_{n \in Z} \{\lambda_{N+k+1}^n\} \right)$$

is a disjoint union of sets σ_n with $|\sigma_n| \leq N+k$ and it can be shown that the sequence of Blascke products $(B_{\sigma_n})_n$ satisfies the Generalized Carleson condition and hence that Λ_a^+ is $(N+k)$ -Carleson. \square

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